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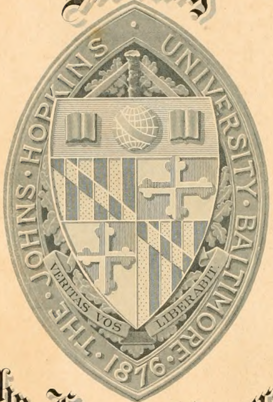
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The Congruences formed by the Tangents to the lines of Curvature of a given Surface.

Dissertation.

Presented to the Board of University studies of the Johns Hopkins University for the degree of Doctor of Philosophy

by

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Baltimore.

1896.

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Introduction.

The theory of congruences is due to Monge, who investigated the fundamental properties (Mémoire de l'Académie des Sciences, 1781), and also the congruence of the normals to a given surface (Application d'Analyse et de Géométrie, théorie des déblais et des remblais). Among the many writers on the subject are Hamilton (Theory of Rays, Irish Transactions, Vol. 15, 16, 17), Kummer (Crelle, Vol. 57), Plücker (Neue Geometrie des Raumes, Leipzig, 1869), Klein (Mathematische Annalen, Vol. 2 and 5), Picard (Thèse), Koenigs (Thèse 1882), Ribaucour (Mémoir couronné par l'Académie de Belgique), Darboux, Clebsch, and Cayley.

Among the many well known proper-

the two surfaces of congruence - may be noticed the following: the surface of centres, which is the focal surface of the congruence of normals to the surface along the lines of curvature. The tangency of the curves of the congruence to the focal surface at the focal points; the two developables which pass through each line of the congruence and are formed of the lines of the congruence; the conjugate system of lines on the surface formed by the two series of developables, of which one set of conjugate lines is composed of the edges of regression of one family of developables and the other of the curves of contact of the developables of the other family; and that knowing a con-

jugate system on a surface (S) we are able in general to obtain an unlimited series of surfaces on which we also know a conjugate system. Taking this last property as a foundation I have at the suggestion and under the direction of Professor Craig made this investigation.

In this investigation I have given the principal formulae necessary for the study of the focal surfaces of the congruences formed by the tangents to the lines of curvature of a given surface (S) and have shown certain properties of these surfaces in special cases, and have given a simple example of such a congruence in taking as the initial surface (S) an Ellipsoid.

I wish to avail myself of this oppor-

tunity to express to Professors Craig and Franklin my appreciation of the kindness and courtesy which they have always shown me and also for the interest which they have manifested in my work during my connection with the Johns Hopkins University.

I.

Let the given surface be denoted by (S) and let u and v be the parameters of the lines of curvature on (S). The letters A, B, C, E, F and G have their usual significance in Gauss's notation. Let $P(x, y, z)$ be any point on the surface (S); ρ_1, ρ_2 the principal radii of curvature corresponding to the u line ($v = \text{const}$) and the v line ($u = \text{const}$) respectively, r_1, r_2 the radii of curvature of the u line and v line respec-

tively; R_2, R_1 the radii of geodesic curvature of the *u* line and *v* line respectively, R_2 being drawn in the positive direction of the tangent to the *v* line and R_1 being drawn in the positive direction of the tangent to the *u* line; τ_1, τ_2 the radii of torsion of the *u* line and *v* line respectively, ϖ_1 the angle between the normal to the surface and the principal normal to the *u* line and ϖ_2 the angle between the normal to the surface and the principal normal to the *v* line.

Let (S_1) denote the one nappe of the focal surface of the congruence formed by the tangents to the *u* lines, i.e. the locus of the extremity of R_1, and (S_2) the one nappe of the focal surface formed by the tangents to the *v* lines i.e. the locus

of the extremity of R_2. As is well known, (S) forms the other nappe of the focal surface in each case. Let $P(X_1, Y_1, Z_1)$ and $P(X_2, Y_2, Z_2)$ be the points on (S_1) and (S_2) corresponding to $P(x, y, z)$ on (S).

For convenience in future reference I shall write a table of formulae for the different direction cosines and their derivatives, also the formulae connecting $R_1, R_2; \rho_1, \rho_2; h_1, h_2; \tau_1, \tau_2; \varpi_1, \varpi_2$; E, F and G. Let α, β, γ be the direction cosines of the normal to the surface (S); α', β', γ' the direction cosines of the tangent to the u line; α', β', γ' the direction cosines of the tangent to the v line; a_1, b_1, c_1 the direction cosines of the principal normal to the u line; $\alpha_1, \beta_1, \gamma_1$ the direction cosines of the binormal to the u line; a_2, b_2, c_2 the direction cosines of the principal normal to the v line; $\alpha_2, \beta_2, \gamma_2$ the direction cosines

of the binormal to the $\underline{v}$ line. We have now for the Table I

$$(1)\quad \begin{cases} \dfrac{1}{R_1} = -\dfrac{1}{2G\sqrt{E}}\cdot\dfrac{\partial G}{\partial u} = -\dfrac{1}{\sqrt{EG}}\cdot\dfrac{\partial\sqrt{G}}{\partial u}, \\[2ex] \dfrac{1}{R_2} = -\dfrac{1}{2E\sqrt{G}}\cdot\dfrac{\partial E}{\partial v} = -\dfrac{1}{\sqrt{EG}}\cdot\dfrac{\partial\sqrt{E}}{\partial v}; \end{cases}$$

$$(2)\quad \begin{cases} \dfrac{1}{h_1^2} = \dfrac{1}{\rho_1^2} + \dfrac{1}{R_2^2}, \\[2ex] \dfrac{1}{h_2^2} = \dfrac{1}{\rho_2^2} + \dfrac{1}{R_1^2}; \end{cases}$$

$$(3)\quad \begin{cases} h_1 = \rho_1 \cos\varpi_1 = -R_2 \sin\varpi_1, \\[1ex] h_2 = \rho_2 \cos\varpi_2 = -R_1 \sin\varpi_2; \end{cases}$$

$$(4)\quad \begin{cases} \tan\varpi_1 = \dfrac{\rho_2}{\rho_2-\rho_1}\cdot\dfrac{1}{\sqrt{G}}\cdot\dfrac{\partial\rho_1}{\partial v} = -\dfrac{\rho_1}{R_2}, \\[2ex] \tan\varpi_2 = \dfrac{\rho_1}{\rho_1-\rho_2}\cdot\dfrac{1}{\sqrt{E}}\cdot\dfrac{\partial\rho_2}{\partial u} = -\dfrac{\rho_2}{R_1}; \end{cases}$$

$$(5)\quad \begin{cases} \dfrac{1}{\tau_1} = \dfrac{\partial\varpi_1}{\sqrt{E}\,\partial u} = \dfrac{h_1^2}{\sqrt{E}\rho_1 R_2}\left[\dfrac{1}{R_2}\cdot\dfrac{\partial R_2}{\partial u} - \dfrac{1}{\rho_1}\cdot\dfrac{\partial\rho_1}{\partial u}\right], \\[2ex] \dfrac{1}{\tau_2} = \dfrac{\partial\varpi_2}{\sqrt{G}\,\partial v} = \dfrac{h_2^2}{\sqrt{G}\rho_2 R_1}\left[\dfrac{1}{R_1}\cdot\dfrac{\partial R_1}{\partial v} - \dfrac{1}{\rho_2}\cdot\dfrac{\partial\rho_2}{\partial v}\right]; \end{cases}$$

$$(6)\quad \begin{cases} \dfrac{\partial\rho_1}{\partial v} = \rho_1\left(1-\dfrac{\rho_1}{\rho_2}\right)\dfrac{1}{2E}\cdot\dfrac{\partial E}{\partial v} = -\rho_1\left(1-\dfrac{\rho_1}{\rho_2}\right)\dfrac{\sqrt{G}}{R_2}, \\[2ex] \dfrac{\partial\rho_2}{\partial u} = \rho_2\left(1-\dfrac{\rho_2}{\rho_1}\right)\dfrac{1}{2G}\cdot\dfrac{\partial G}{\partial u} = -\rho_2\left(1-\dfrac{\rho_2}{\rho_1}\right)\dfrac{\sqrt{E}}{R_1}; \end{cases}$$

$$(7)\quad \begin{cases} \alpha = \dfrac{A}{\sqrt{EG}} = \dfrac{1}{\sqrt{EG}}\cdot\dfrac{\partial(y,z)}{\partial(u,v)}, \\[2ex] \beta = \dfrac{B}{\sqrt{EG}} = \dfrac{1}{\sqrt{EG}}\cdot\dfrac{\partial(z,x)}{\partial(u,v)}, \\[2ex] \gamma = \dfrac{C}{\sqrt{EG}} = \dfrac{1}{\sqrt{EG}}\cdot\dfrac{\partial(x,y)}{\partial(u,v)}; \end{cases}$$

$$(8)\quad \begin{cases} \alpha'' = \dfrac{1}{\sqrt{G}}\,\dfrac{\partial x}{\partial v}, \\[2ex] \beta'' = \dfrac{1}{\sqrt{G}}\,\dfrac{\partial y}{\partial v}, \\[2ex] \gamma'' = \dfrac{1}{\sqrt{G}}\,\dfrac{\partial z}{\partial v}, \end{cases}$$

$$(9)\quad \begin{cases} \alpha' = \dfrac{1}{\sqrt{E}} \cdot \dfrac{\partial x}{\partial u}, \\ \beta' = \dfrac{1}{\sqrt{E}} \cdot \dfrac{\partial y}{\partial u}, \\ \gamma' = \dfrac{1}{\sqrt{E}} \cdot \dfrac{\partial z}{\partial u}; \end{cases}$$

$$(10)\quad \begin{cases} a_1 = h_1\left(\dfrac{\alpha}{\rho_1} + \dfrac{\alpha''}{R_2}\right), \\ b_1 = h_1\left(\dfrac{\beta}{\rho_1} + \dfrac{\beta''}{R_2}\right), \\ c_1 = h_1\left(\dfrac{\gamma}{\rho_1} + \dfrac{\gamma''}{R_2}\right); \end{cases}$$

$$(11)\quad \begin{cases} a_2 = h_2\left(\dfrac{\alpha}{\rho_2} + \dfrac{\alpha'}{R_1}\right), \\ b_2 = h_2\left(\dfrac{\beta}{\rho_2} + \dfrac{\beta'}{R_1}\right), \\ c_2 = h_2\left(\dfrac{\gamma}{\rho_2} + \dfrac{\gamma'}{R_1}\right); \end{cases}$$

$$(12)\quad \begin{cases} \alpha_1 = h_1\left(\dfrac{\alpha'}{\rho_1} - \dfrac{\alpha}{R_2}\right), \\ \beta_1 = h_1\left(\dfrac{\beta''}{\rho_1} - \dfrac{\beta}{R_2}\right), \\ \gamma_1 = h_1\left(\dfrac{\gamma''}{\rho_1} - \dfrac{\gamma}{R_2}\right); \end{cases}$$

$$(13)\quad \begin{cases} \alpha_2 = h_2\left(\dfrac{\alpha'}{\rho_2} - \dfrac{\alpha}{R_1}\right), \\ \beta_2 = h_2\left(\dfrac{\beta'}{\rho_2} - \dfrac{\beta}{R_1}\right), \\ \gamma_2 = h_2\left(\dfrac{\gamma'}{\rho_2} - \dfrac{\gamma}{R_1}\right); \end{cases}$$

$$(14)\quad \begin{cases} \dfrac{\partial \alpha}{\partial u} = -\dfrac{\sqrt{E}}{\rho_1}\,\alpha', \\ \dfrac{\partial \beta}{\partial u} = -\dfrac{\sqrt{E}}{\rho_1}\,\beta', \\ \dfrac{\partial \gamma}{\partial u} = -\dfrac{\sqrt{E}}{\rho_1}\,\gamma' \end{cases}$$

$$(14')\quad\begin{cases}\dfrac{\partial\alpha}{\partial v} = -\dfrac{\sqrt{G}}{\rho_2}\,\alpha', \\[2ex] \dfrac{\partial\beta}{\partial v} = -\dfrac{\sqrt{G}}{\rho_2}\,\beta''; \\[2ex] \dfrac{\partial\gamma}{\partial v} - \dfrac{\sqrt{G}}{\rho_2}\,\gamma''; \end{cases}$$

$$(15)\quad\begin{cases}\dfrac{\partial\alpha'}{\partial u} = \sqrt{E}\left(\dfrac{\alpha}{\rho_1} + \dfrac{\alpha''}{R_2}\right), \\[2ex] \dfrac{\partial\beta'}{\partial u} = \sqrt{E}\left(\dfrac{\beta}{\rho_1} + \dfrac{\beta''}{R_2}\right), \\[2ex] \dfrac{\partial\gamma'}{\partial u} = \sqrt{E}\left(\dfrac{\gamma}{\rho_1} + \dfrac{\gamma''}{R_2}\right); \end{cases}$$

$$(15')\quad\begin{cases}\dfrac{\partial\alpha'}{\partial v} = -\dfrac{\sqrt{G}}{R_1}\,\alpha'', \\[2ex] \dfrac{\partial\beta'}{\partial v} = -\dfrac{\sqrt{G}}{R_1}\,\beta'', \\[2ex] \dfrac{\partial\gamma'}{\partial v} = -\dfrac{\sqrt{G}}{R_1}\,\gamma'', \end{cases}$$

$$(16)\quad\begin{cases}\dfrac{\partial\alpha''}{\partial u} = -\dfrac{\sqrt{E}}{R_2}\,\alpha', \\[2ex] \dfrac{\partial\beta''}{\partial u} = -\dfrac{\sqrt{E}}{R_2}\,\beta', \\[2ex] \dfrac{\partial\gamma''}{\partial u} = -\dfrac{\sqrt{E}}{R_2}\,\gamma'; \end{cases}$$

$$(16')\quad\begin{cases}\dfrac{\partial\alpha''}{\partial v} = \sqrt{G}\left(\dfrac{\alpha}{\rho_2} + \dfrac{\alpha'}{R_1}\right), \\[2ex] \dfrac{\partial\beta''}{\partial v} = \sqrt{G}\left(\dfrac{\beta}{\rho_2} + \dfrac{\beta'}{R_1}\right), \\[2ex] \dfrac{\partial\gamma''}{\partial v} = \sqrt{G}\left(\dfrac{\gamma}{\rho_2} + \dfrac{\gamma'}{R_1}\right). \end{cases}$$

We have now for the coordinates of a point on the surfaces (S_1) and (S_2) respectively

$$(17)\quad \begin{cases} X_1 = x + R_1 \frac{1}{\sqrt{E}} \cdot \frac{\partial x}{\partial u} = x + R_1 \alpha'; \\ Y_1 = y + R_1 \frac{1}{\sqrt{E}} \cdot \frac{\partial y}{\partial u} = y + R_1 \beta'; \\ Z_1 = z + R_1 \frac{1}{\sqrt{E}} \frac{\partial z}{\partial u} = z + R_1 \gamma'; \end{cases}$$

$$(18)\quad \begin{cases} X_2 = x + R_2 \frac{1}{\sqrt{G}} \frac{\partial x}{\partial v} = x + R_2 \alpha''; \\ Y_2 = y + R_2 \frac{1}{\sqrt{G}} \cdot \frac{\partial y}{\partial v} = y + R_2 \beta''; \\ Z_2 = z + R_2 \frac{1}{\sqrt{G}} \cdot \frac{\partial z}{\partial v} = z + R_2 \gamma''. \end{cases}$$

We shall now proceed to calculate the fundamental coefficients E_1, F_1, G_1, L_1, N_1 for the surface (S_1); those for the surface (S_2) may be derived from those of (S_1) by certain obvious changes of letters and suffixes. As the lines on the surfaces (S_1) and (S_2) corresponding to the $\underline{u}$ and $\underline{v}$ lines on the surface (S) are conjugate we have

$$(19)\qquad M_1 = M_2 = 0.$$

On differentiating equations (17) using expressions (2) (8), (9) (14) (15) and (16), Part I; and writing for brevity

(20) $$U = 1 + \frac{1}{\sqrt{E}}\frac{\partial R_1}{\partial u}$$

we have

(21) $$\left\{\begin{aligned} \frac{\partial X_1}{\partial u} &= \sqrt{E}\left[U\alpha + R_1\left\{\frac{\alpha}{\rho_1} + \frac{\alpha''}{R_2}\right\}\right], \\ \frac{\partial Y_1}{\partial u} &= \sqrt{E}\left[U\beta' + R_1\left\{\frac{\beta}{\rho_1} + \frac{\beta''}{R_2}\right\}\right], \\ \frac{\partial Z_1}{\partial u} &= \sqrt{E}\left[U\gamma' + R_1\left\{\frac{\gamma}{\rho_1} + \frac{\gamma''}{R_2}\right\}\right]; \end{aligned}\right.$$

(22) $$\left\{\begin{aligned} \frac{\partial X_1}{\partial v} &= \alpha'\frac{\partial R_1}{\partial v}, \\ \frac{\partial Y_1}{\partial v} &= \beta'\frac{\partial R_1}{\partial v}, \\ \frac{\partial Z_1}{\partial v} &= \gamma'\frac{\partial R_1}{\partial v}; \end{aligned}\right.$$

(23) $$\left\{\begin{aligned} \frac{\partial^2 X_1}{\partial u^2} &= \left[\frac{\alpha'}{2\sqrt{E}}\cdot\frac{\partial E}{\partial u} + E\left\{\frac{\alpha}{\rho_1} + \frac{\alpha}{R_2}\right\}\right]U + \sqrt{E}\left[\alpha\frac{\partial U}{\partial u} + \left\{\frac{\alpha}{\rho} + \frac{\alpha}{R_2}\right\}\left\{\frac{1}{2E}\frac{\partial E}{\partial u}R_1 + \frac{\partial R_1}{\partial u}\right\}\right. \\ &\qquad \left. - R_1\left\{\frac{\sqrt{E}}{R_1^2}\alpha' + \frac{\alpha}{\rho_1^2}\frac{\partial \rho_1}{\partial u} + \frac{\alpha''}{R_2^2}\frac{\partial R_2}{\partial u}\right\}\right], \\ \frac{\partial^2 Y_1}{\partial u^2} &= \left[\frac{\beta}{2\sqrt{E}}\cdot\frac{\partial E}{\partial u} + E\left\{\frac{\beta}{\rho_1} + \frac{\beta}{R_2}\right\}\right]U + \sqrt{E}\left[\beta\frac{\partial U}{\partial u} + \left\{\frac{\beta}{\rho_1} + \frac{\beta'}{R_2}\right\}\left\{\frac{1}{2E}\frac{\partial E}{\partial u}R_1 + \frac{\partial R_1}{\partial u}\right\}\right. \\ &\qquad \left. - R_1\left\{\frac{\sqrt{E}}{R_1^2}\beta' + \frac{\beta}{\rho_1^2}\frac{\partial \rho_1}{\partial u} + \frac{\beta''}{R_2^2}\frac{\partial R_2}{\partial u}\right\}\right], \\ \frac{\partial^2 Z_1}{\partial u^2} &= \left[\frac{\gamma'}{2\sqrt{E}}\frac{\partial E}{\partial u} + E\left\{\frac{\gamma}{\rho_1} + \frac{\gamma}{R_2}\right\}\right]U + \sqrt{E}\left[\gamma\frac{\partial U}{\partial u} + \left\{\frac{\gamma}{\rho_1} + \frac{\gamma}{R_2}\right\}\left\{\frac{1}{2E}\frac{\partial E}{\partial u}R_1 + \frac{\partial R_1}{\partial u}\right\}\right. \\ &\qquad \left. - R_1\left\{\frac{\sqrt{E}}{R_1^2}\gamma' + \frac{\gamma}{\rho_1^2}\frac{\partial \rho_1}{\partial u} + \frac{\gamma''}{R_2^2}\frac{\partial R_2}{\partial u}\right\}\right]; \end{aligned}\right.$$

(24) $$\left\{\begin{aligned} \frac{\partial^2 X_1}{\partial v^2} &= \frac{\partial^2 R_1}{\partial v^2}\alpha' - \frac{\sqrt{G}}{R_1}\cdot\frac{\partial R_1}{\partial v}\alpha, \\ \frac{\partial^2 Y_1}{\partial v^2} &= \frac{\partial^2 R_1}{\partial v^2}\beta' - \frac{\sqrt{G}}{R_1}\cdot\frac{\partial R_1}{\partial v}\beta, \\ \frac{\partial^2 Z_1}{\partial v^2} &= \frac{\partial^2 R_1}{\partial v^2}\gamma' - \frac{\sqrt{G}}{R_1}\frac{\partial R_1}{\partial v}\gamma. \end{aligned}\right.$$

$$(25)\qquad \begin{cases} \dfrac{\partial^2 X_1}{\partial u\,\partial v} = \dfrac{\partial^2 R_1}{\partial u\,\partial v}\alpha' + \sqrt{E}\,\dfrac{\partial R_1}{\partial v}\left\{\dfrac{\alpha}{\rho_1} + \dfrac{\alpha''}{R_2}\right\}, \\ \dfrac{\partial^2 Y_1}{\partial u\,\partial v} = \dfrac{\partial^2 R_1}{\partial u\,\partial v}\beta' + \sqrt{E}\,\dfrac{\partial R_1}{\partial v}\left\{\dfrac{\beta}{\rho_1} + \dfrac{\beta''}{R_2}\right\}, \\ \dfrac{\partial^2 Z_1}{\partial u\,\partial v} = \dfrac{\partial^2 R_1}{\partial u\,\partial v}\gamma' + \sqrt{E}\,\dfrac{\partial R_1}{\partial v}\left\{\dfrac{\gamma}{\rho_1} + \dfrac{\gamma''}{R_2}\right\}. \end{cases}$$

On using the equations (21) and (22) we find the following expressions for A_1, B_1, C_1 and H_1,

$$(26)\qquad \begin{cases} A_1 = \dfrac{\partial(Y_1, Z_1)}{\partial(u, v)} = \sqrt{E}\,R_1\dfrac{\partial R_1}{\partial v}\left\{\dfrac{\alpha}{\rho_1} - \dfrac{\alpha}{R_2}\right\}, \\ B_1 = \dfrac{\partial(Z_1, X_1)}{\partial(u, v)} = \sqrt{E}\,R_1\dfrac{\partial R_1}{\partial v}\left\{\dfrac{\beta''}{\rho_1} - \dfrac{\beta}{R_2}\right\}, \\ C_1 = \dfrac{\partial(X_1, Y_1)}{\partial(u, v)} = \sqrt{E}\,R_1\dfrac{\partial R_1}{\partial v}\left\{\dfrac{\gamma'}{\rho_1} - \dfrac{\gamma}{R_2}\right\}; \end{cases}$$

$$(27)\qquad H_1^2 = E_1G_1 - F_1^2 = A_1^2 + B_1^2 + C_1^2 = \frac{E R_1^2}{h_1^2}\left(\frac{\partial R_1}{\partial v}\right)^2.$$

We also find on using equations (21) and (22) the following expressions for E_1, F_1 and G_1.

$$(28)\qquad \begin{cases} E_1 = \sum\left(\dfrac{\partial X_1}{\partial u}\right)^2 = \dfrac{E}{h_1^2}\left\{h_1^2U^2 + R_1^2\right\}, \\ F_1 = \sum \dfrac{\partial X_1}{\partial u}\cdot\dfrac{\partial X_1}{\partial v} = \sqrt{E}\,U\,\dfrac{\partial R_1}{\partial v}, \\ G_1 = \sum\left(\dfrac{\partial X_1}{\partial v}\right)^2 = \left(\dfrac{\partial R_1}{\partial v}\right)^2. \end{cases}$$

If now we denote for the instant the direction cosines of the normal to the surface (S_1) by a, b, c we shall have on using equations (26) and (27)

$$(29)\qquad \begin{cases} a = \dfrac{A_1}{H_1} = k_1\left\{\dfrac{\alpha}{\rho_1} - \dfrac{\alpha}{R_2}\right\}, \\ b = \dfrac{B_1}{H_1} = k_1\left\{\dfrac{\beta}{\rho_1} - \dfrac{\beta}{R_2}\right\}, \\ c = \dfrac{C_1}{H_1} = k_1\left\{\dfrac{\gamma}{\rho_1} - \dfrac{\gamma}{R_2}\right\}; \end{cases}$$

and on comparing this with equations (12) Table I we see that

$$(30)\qquad a = \alpha_1,\ b = \beta_1,\ c = \gamma_1,$$

that is that the normal to the surface (S_1) at the point $P(X_1, Y_1, Z_1)$ is parallel to the binormal to the curve $\underline{u}$ on the surface (S) at the point $P(x, y, z)$ and consequently that the tangent plane to the surface (S_1) at the point $P(X_1, Y_1, Z_1)$ is the osculating plane of the curve $\underline{u}$ at the point $P(x, y, z)$ on the surface (S), which indicates that the surface (S_1) may be considered as the envelope of the osculating planes of the $\underline{u}$ curve at each point $P(x, y, z)$ on the surface (S).

The fundamental coefficients L, $M(=0)$, and

N of the surface (S) are defined by the equations

(31) $$L = \sum \alpha \frac{\partial^2 x}{\partial u^2}, \quad M = \sum \alpha \frac{\partial^2 x}{\partial u \partial v}, \quad N = \sum \alpha \frac{\partial^2 x}{\partial v^2}.$$

We have then for the corresponding elements on the surface (S_1) the equations

(32) $$L_1 = \sum \alpha_1 \frac{\partial^2 X}{\partial u^2}, \quad M_1 = \sum \alpha_1 \frac{\partial^2 X}{\partial u \partial v}, \quad N_1 = \sum \alpha_1 \frac{\partial^2 X}{\partial v^2},$$

which become, on using equations (23), (24), (25), and equation (3), Table I

(33) $$\begin{cases} L_1 = -\dfrac{h_1 R_1 \sqrt{E}}{\rho_1 R_2}\left\{\dfrac{1}{R_2}\cdot\dfrac{\partial R_2}{\partial u} - \dfrac{1}{\rho_1}\dfrac{\partial \rho_1}{\partial u}\right\} = -\dfrac{E R_1}{h_1 \tau_1}, \\ M_1 = 0, \\ N_1 = -\dfrac{\sqrt{G}\, h_1}{R_1 \rho_1}\cdot\dfrac{\partial R_1}{\partial v}; \end{cases}$$

(34) $$L_1 N_1 = \frac{E\sqrt{G}}{\rho_1 \tau_1}\cdot\frac{\partial R_1}{\partial v}.$$

On interchanging the parameters u and v, which is equivalent to interchanging E and G, and the suffixes of the other letters, we have, as has been before stated, the fundamental coefficients for the surface (S_2); and furthermore all properties and theorems derived for the sur-

face (S_1), after the above interchange, will be true for the surface (S_2) in general.

If the point $P(x,y,z)$ moves along a line of curvature on the surface (S) the extremity of R_1 or R_2 traces out a curve on the surface (S_1) or (S_2) concerning which we have the known theorem (Darboux Théorie générale des surfaces t III, page 121).

The locus of the centres of geodesic curvature of a line of curvature is that one of its evolutes which is the envelope of the normals to the line of curvature lying in the tangent plane to the surface or, what is the same thing, the edge of regression of the developable enveloped by the tangent planes to

The principal plane point in the line of curvature".

The consideration of this theorem and the consequences arising from it will be taken up later but for the present I shall consider the differential equations of the surfaces (S) and (S_1).

If u and v are the parameters of a system of conjugate lines on the surface (S), we know that x, y, z are particular solutions of the partial differential equation

$$(35) \qquad \frac{\partial^2\theta}{\partial u\,\partial v} + a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} = 0,$$

where

$$(36) \qquad \begin{cases} a = -\dfrac{G\frac{\partial E}{\partial v} - F\frac{\partial G}{\partial u}}{2(EG-F^2)}, \\ b = -\dfrac{E\frac{\partial G}{\partial u} - F\frac{\partial E}{\partial v}}{2(EG-F^2)}. \end{cases}$$

or for our particular case, where u and v are the parameters of the lines

of curvature,

$$(37)\qquad \begin{cases} a = -\dfrac{1}{2E}\dfrac{\partial E}{\partial v} = \dfrac{\sqrt{G}}{R_2}, \\ b = -\dfrac{1}{2G}\dfrac{\partial G}{\partial u} = \dfrac{\sqrt{E}}{R_1} \end{cases}$$

Let us denote by h and k the invariants in equation (35), then we have

$$(38)\qquad \begin{cases} h = \dfrac{\partial a}{\partial u} + ab = -\dfrac{\sqrt{G}}{R_2^2}\cdot\dfrac{\partial R_2}{\partial u}, \\ k = \dfrac{\partial b}{\partial v} + ab = -\dfrac{\sqrt{E}}{R_1^2}\cdot\dfrac{\partial R_1}{\partial v}. \end{cases}$$

To find the transformed equation satisfied by X_1, Y_1, Z_1 observe that

$$(39)\qquad \begin{cases} X_1 = x + \dfrac{1}{b}\cdot\dfrac{\partial x}{\partial u}, \\ Y_1 = y + \dfrac{1}{b}\cdot\dfrac{\partial y}{\partial u}, \\ Z_1 = z + \dfrac{1}{b}\dfrac{\partial z}{\partial u}. \end{cases}$$

The formula of transformation can therefore be written

$$(40)\qquad \theta_1 = \theta + \frac{1}{b}\frac{\partial\theta}{\partial u}.$$

If now we differentiate equation (40) for v and make certain reductions, we have

$$(41)\qquad \frac{\partial\theta_1}{\partial v} = -\frac{k}{b^2}\frac{\partial\theta}{\partial u},$$

from which we derive

$$(42) \qquad \sigma = \theta - \frac{b}{k}\frac{\partial \sigma}{\partial v};$$

now differentiating equation (42) for u, substituting the values of b and k drawn from equations (37) and (38) and reducing we have for the differential equation of the surface (S_1) the equation

$$(43) \qquad \frac{\partial^2 \sigma}{\partial u \partial v} - \frac{1}{R_1}\frac{\partial R_1}{\partial v}\cdot\frac{\partial \sigma}{\partial u} + \left\{ \frac{\sqrt{E}}{R_1} U - \frac{\partial}{\partial u} \log \frac{\partial R_1}{\partial v} \right\} \frac{\partial \sigma}{\partial v} = 0,$$

where as before

$$(44) \qquad \bar{U} = 1 + \frac{1}{\sqrt{E}}\frac{\partial R_1}{\partial u}.$$

We know that the equation satisfied by the coordinates $X_1 Y_1 Z_1$ should be of the form (Darboux, Tome II, Page 12; Tome I, Page 122)

$$(45) \qquad \frac{\partial^2 \sigma}{\partial u \partial v} - \frac{G_1 \frac{\partial E_1}{\partial v} - F_1 \frac{\partial G_1}{\partial u}}{2(E_1 G_1 - F_1^2)} \frac{\partial \sigma}{\partial u} - \frac{E_1 \frac{\partial G_1}{\partial u} - F_1 \frac{\partial E_1}{\partial v}}{2(E_1 G_1 - F_1^2)} \frac{\partial \sigma}{\partial v} = 0.$$

The identity of equations (43) and (45) is easily verified. We have only to compute the values of $\frac{\partial E}{\partial v}$ and $\frac{\partial G_1}{\partial u}$. The latter is found at once.

$$(46) \qquad G_1 = \left(\frac{\partial R_1}{\partial v}\right)^2, \quad \frac{\partial G_1}{\partial u} = 2\frac{\partial R_1}{\partial v}\cdot\frac{\partial^2 R_1}{\partial u \partial v}.$$

To compute the value of $\frac{\partial E_1}{\partial v}$ it is convenient to introduce Liouville's expression for the measure of curvature. This in the present case where u and v are the parameters of the lines of curvature on the surface (S), is

$$(47) \qquad \frac{1}{\rho_1\rho_2} = \frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\frac{\sqrt{G}}{R_1} + \frac{\partial}{\partial v}\frac{\sqrt{E}}{R_2}\right] = -\left\{\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{\sqrt{E}R_1^2}\cdot\frac{\partial R_1}{\partial u} + \frac{1}{\sqrt{G}R_2^2}\frac{\partial R_2}{\partial v}\right\}.$$

If now we divide the first of equations (6), Table I, by ρ_1^4 we have on substituting for $\frac{1}{\rho_1\rho_2}$ its value drawn from equation (47)

$$(48) \qquad \frac{1}{\rho_1^3}\frac{\partial \rho_1}{\partial v} = -\left\{\frac{\sqrt{G}}{\rho_1^2 R_2} + \frac{\sqrt{G}}{R_1^2 R_2} + \frac{\sqrt{G}}{R_2^3} + \frac{\sqrt{G}}{\sqrt{E}}\cdot\frac{1}{R_1^2 R_2}\cdot\frac{\partial R_1}{\partial u} - \frac{1}{R_2^3}\frac{\partial R_2}{\partial v}\right\}.$$

Now differentiating the first of equations (2), Table I, with respect to v we have

$$(49) \qquad \frac{1}{h_1^3}\frac{\partial h_1}{\partial v} = \frac{1}{\rho_1^3}\frac{\partial \rho_1}{\partial v} + \frac{1}{R_2^3}\cdot\frac{\partial R_2}{\partial v}.$$

We now have

$$(50) \qquad E_1 = \frac{E}{h_1^2}\left\{h_1^2 U^2 + R_1^2\right\},$$

$$(51) \qquad E_1 G_1 - F_1^2 = \frac{E R_1^2}{h_1^2}\left(\frac{\partial R_1}{\partial v}\right)^2,$$

Upon differentiating equation (50) with

respect to v and using equations (47), (48), and (49) we find after easy reductions

$$(52)\qquad \frac{\partial E_1}{\partial v} = \frac{2ER_1}{h_1^2}\cdot\frac{\partial R_1}{\partial v} + 2\sqrt{E}\,U\frac{\partial^2 R_1}{\partial u\,\partial v}.$$

Using now equations (50) (51) (52) and the values of E_1, F_1, and G_1 we find that

$$(53)\qquad \begin{cases} -\dfrac{G_1\frac{\partial E_1}{\partial v} - F_1\frac{\partial G_1}{\partial u}}{2(E_1G_1 - F_1^2)} = -\dfrac{1}{R_1}\dfrac{\partial R_1}{\partial v}, \\[2ex] -\dfrac{E_1\frac{\partial G_1}{\partial u} - F_1\frac{\partial E_1}{\partial v}}{2(E_1G_1 - F_1^2)} = \dfrac{\sqrt{E}}{R_1}U - \dfrac{\partial}{\partial u}\log\dfrac{\partial R_1}{\partial v}, \end{cases}$$

which shows the identity of the coefficients of equations (43) and (45).

I shall now consider the invariants h and k of the differential equation of the surface (S) and see what the surface (S) becomes when they are equal.

From equation (38) we have under the hypothesis

$$(54)\qquad \frac{\sqrt{G}}{R_2^2}\frac{\partial R_2}{\partial u} = \frac{\sqrt{E}}{R_1^2}\frac{\partial R_1}{\partial v}.$$

I shall make before considering the general case two other hypotheses

and see what happens in each case. First I shall suppose

(55) $$\frac{\partial^2 G}{\partial u^2} = 0, \quad \frac{\partial^2 E}{\partial v^2} = 0,$$

then we shall have

(56) $$\begin{cases} E = v\,\varphi_1(u) + \varphi_2(u), \\ G = u\,\psi_1(v) + \psi_2(v), \end{cases}$$

and furthermore

(57) $$\begin{cases} \dfrac{1}{R_1} = -\dfrac{\psi_1(v)}{2\{u\psi_1(v)+\psi_2(v)\}\{v\varphi_1(u)+\varphi_2(u)\}^{1/2}}, \\ \dfrac{1}{R_2} = -\dfrac{\varphi_1(u)}{2\{v\varphi_1(u)+\varphi_2(u)\}\{u\psi_1(v)+\psi_2(v)\}^{1/2}}. \end{cases}$$

The equation of condition (54) now becomes

$$\frac{\partial}{\partial u}\cdot\frac{\varphi_1(u)}{v\varphi_1(u)+\varphi_2(u)} = \frac{\partial}{\partial v}\,\frac{\psi_1(v)}{u\psi_1(v)+\psi_2(v)},$$

$$\frac{\partial}{\partial u}\cdot\frac{\sqrt{G}}{R_2} = \frac{\partial}{\partial v}\,\frac{\sqrt{E}}{R_1},$$

$$\frac{\partial}{\partial u}\cdot\frac{\partial \log E}{\partial v} = \frac{\partial}{\partial v}\cdot\frac{\partial \log G}{\partial u},$$

(58) $$\frac{\partial^2 \log E}{\partial u\,\partial v} = \frac{\partial^2 \log G}{\partial u\,\partial v},$$

(59) $$\therefore\quad E = G\,\tau(u)\,\sigma(v).$$

The element of length becomes now

(60) $$ds^2 = T(u_1,v_1)\left[du_1^2+dv_1^2\right],$$

which indicates a surface on which $\underline{u}$ and $\underline{v}$ are isothermal lines.

Now I shall suppose

$$E = \frac{\psi^2(v)}{G}\left(\frac{\partial G}{\partial u}\right)^2; \tag{61}$$

in this case we have

$$\left\{\begin{aligned} \frac{1}{R_1} &= -\frac{1}{2\sqrt{G}\,\psi^2(v)}, \\ \frac{1}{R_2} &= -\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\log\left\{\frac{\psi(v)}{\sqrt{G}}\frac{\partial G}{\partial u}\right\}; \end{aligned}\right. \tag{62}$$

and our equation of condition (54) becomes after some reductions

$$\frac{\partial^2}{\partial u \partial v}\log\frac{\frac{\partial G}{\partial u}}{G} = 0; \tag{63}$$

which gives on integration

$$\left\{\begin{aligned} G &= \tau(v)\,e^{\sigma(u)\varphi(v)}, \\ E &= \psi^2(v)\,\varphi^2(v)\,\sigma^2(u)\,G; \end{aligned}\right. \tag{64}$$

and the element of length becomes

$$ds^2 = T_1(u_1, v_1)\left[du_1^2 + dv_1^2\right], \tag{65}$$

which again indicates a surface on which $\underline{u}$ and $\underline{v}$ are isothermal lines.

Now I shall consider the equation of condition (54) without any further hy-

pothesis and see what the surface becomes in that case. We have

$$\frac{\sqrt{G}}{R_2^2}\frac{\partial R_2}{\partial u} = \frac{\sqrt{E}}{R_1^2}\frac{\partial R_1}{\partial v}, \tag{66}$$

which after substituting the values of R_1, R_2 given in expressions (1), and performing the indicated differentiation and reducing, becomes

$$\frac{1}{E}\frac{\partial^2 E}{\partial u \partial v} - \frac{1}{E^2}\frac{\partial E}{\partial u}\cdot\frac{\partial E}{\partial v} = \frac{1}{G}\frac{\partial^2 G}{\partial u \partial v} - \frac{1}{G^2}\frac{\partial G}{\partial u}\cdot\frac{\partial G}{\partial v}, \tag{67}$$

or

$$\frac{\partial^2 \log E}{\partial u \partial v} = \frac{\partial^2 \log G}{\partial u \partial v}; \tag{68}$$

therefore

$$E = G\,\tau(u)\,\sigma(v). \tag{69}$$

The element of length becomes

$$ds^2 = T_2(u_1, v_1)\left[du_1^2 + dv_1^2\right], \tag{70}$$

which indicates a surface on which $\underline{u}$ and $\underline{v}$ are isothermal lines. Hence we see that in general the equality of the invariants $\underline{h}$ and $\underline{k}$ is a sufficient condition for a surface on which $\underline{u}$ and

v are isothermal lines. Among the surfaces for which the lines of curvature are isothermal, Darboux (Théorie générale des surfaces II, page 240) has indicated the Quadric surfaces, Cyclides, Cones, Cylinders, Surfaces of Revolution, Minimal surfaces, surfaces of constant mean curvature, certain surfaces with plane lines of curvature in a system which depends on an arbitrary parameter, together with those derived from the above by inversion.

On writing

(71) $$r_1 = \frac{1}{\rho}\,;\quad r_2 = \frac{1}{\rho_2}\,;\quad g_1 = \frac{1}{R_1}\,;\quad g_2 = \frac{1}{R_2}\,;$$

and using the following expressions derived from equations (6), Table I,

(72) $$\frac{1}{\rho_1^2}\frac{\partial \rho_1}{\partial v} = -\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\frac{\sqrt{G}}{R_2},$$

(73) $$\frac{1}{\rho_2^2}\frac{\partial \rho_2}{\partial u} = +\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\frac{\sqrt{E}}{R_1},$$

we derive on dividing equation (72) by

equation (73) and substituting from the relations (71) the following expression,

$$\sqrt{E}\,\frac{1}{g_2}\cdot\frac{\partial r_1}{\partial v} + \sqrt{G}\,\frac{1}{g_1}\,\frac{\partial r_2}{\partial u} = 0 \qquad (74)$$

We also have on making in equation (66), the substitutions expressed in relations (71)

$$-\sqrt{E}\,\frac{\partial g_1}{\partial v} + \sqrt{G}\,\frac{\partial g_2}{\partial u} = 0. \qquad (75)$$

If now we eliminate the $\sqrt{E}$ and the $\sqrt{G}$ between equations (74) and (75) we derive for the case where the invariants h and k are equal the following expression connecting the curvature of the principal sections with the geodesic curvature of the u and v lines,

$$\frac{\partial \log g_2}{\partial u}\,\frac{\partial r_1}{\partial v} + \frac{\partial \log g_1}{\partial v}\,\frac{\partial r_2}{\partial u} = 0, \qquad (76)$$

I shall now return to the theorem cited from Darboux on page (15). If the line of curvature v (u = const.) has its radius of geodesic curvature constant, we

[illegible]

(37) $\frac{dR_1}{dv} = 0 . \quad R_1 = f(u).$

In this case we have

(38) [illegible] $= N = 0,$

and the surface (S_1) reduces to a curve. In fact (Darboux, t III, 121) the evolute of a v-line reduces to a point and consequently the line of curvature on the surface (S) lies on a sphere which cuts (S) at right angles. If the same is true for the u lines the surface (S_2) will also reduce to a curve and the evolute of any one u line reduces to a point. If then we have both of the equations

(39) $\frac{dR_1}{dv} = 0, \quad \frac{dR_2}{du} = 0$

satisfied the surfaces (S_1) and (S_2) each reduce to a curve. Concerning the surfaces for which this is true we have

the theorem (loc. cit. 122):

"The only surfaces for which all the lines of curvature have their geodesic curvature constant are the surfaces of revolution, cones, cylinders and the transforms of these by inversion."

For example take the surfaces of revolution given by the formulae

$$(80)\qquad \begin{cases} x = u \sin v, \\ y = u \cos v, \\ z = f(u). \end{cases}$$

We have for the linear element

$$(81)\qquad ds^2 = \left\{1+f'^2(u)\right\} du^2 + u^2 dv^2,$$

which indicates that

$$(82)\qquad E = 1+f'^2(u), \quad G = u^2$$

$$(83)\qquad \frac{1}{R_1} = -\frac{1}{u\sqrt{1+f'^2(u)}}, \quad \frac{1}{R_2} = 0.$$

These cases will in general be excluded from consideration.

In order that the lines u and v may

be lines of curvature on the surfaces (S₁) and (S₂) we must have

(84) $$F_1 = F_2 = 0.$$

Since we have excluded the cases where

(85) $$\frac{\partial R_1}{\partial v} = 0, \quad \frac{\partial R_2}{\partial u} = 0,$$

we must have in order to satisfy equations (84)

(86) $$\begin{cases} 1 + \frac{1}{\sqrt{E}} \cdot \frac{\partial R_1}{\partial u} = 0, \\ 1 + \frac{1}{\sqrt{G}} \cdot \frac{\partial R_2}{\partial v} = 0. \end{cases}$$

or as these may be written

(87) $$\begin{cases} -\frac{\sqrt{E}}{R_1} = \frac{1}{R_1} \cdot \frac{\partial R_1}{\partial u}. \\ -\frac{\sqrt{G}}{R_2} = \frac{1}{R_2} \cdot \frac{\partial R_2}{\partial v}. \end{cases}$$

Before considering the equations (87) it is necessary to give two expressions for the measure of curvature of the surface (S) at the point $P(x, y, z)$. These are those given by Gauss and by Liouville respectively in accordance with the fact

that u and v are the parameters of the lines of curvature of the surface (S). Gauss's formula is

(88) $$K = \frac{LN}{EG} = \frac{1}{2EG}\left\{\frac{1}{2G}\frac{\partial E}{\partial v}\frac{\partial G}{\partial v} + \frac{1}{2E}\left(\frac{\partial E}{\partial v}\right)^2 - \frac{\partial^2 E}{\partial v^2} + \frac{1}{2E}\frac{\partial G}{\partial u}\frac{\partial E}{\partial u} + \frac{1}{2G}\left(\frac{\partial G}{\partial u}\right)^2 - \frac{\partial^2 G}{\partial u^2}\right\}.$$

Liouville's formula is

(89) $$K = \frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}\frac{\sqrt{G}}{R_1} + \frac{\partial}{\partial v}\frac{\sqrt{E}}{R_2}\right\} = -\frac{1}{R_1^2}\left\{1 + \frac{1}{\sqrt{E}}\frac{\partial R_1}{\partial u}\right\} - \frac{1}{R_2^2}\left\{1 + \frac{1}{\sqrt{G}}\frac{\partial R_2}{\partial v}\right\}.$$

If both of equations (80) are satisfied, we have $K = \frac{1}{\rho_1\rho_2} = 0$ from equation (89). Expanding the equations (80) we have

(90) $$\frac{1}{2E}\frac{\partial E}{\partial u}\cdot\frac{\partial G}{\partial u} + \frac{1}{2G}\left(\frac{\partial G}{\partial u}\right)^2 - \frac{\partial^2 G}{\partial u^2} = 0,$$

(91) $$\frac{1}{2G}\frac{\partial E}{\partial v}\frac{\partial G}{\partial v} + \frac{1}{2E}\left(\frac{\partial E}{\partial v}\right)^2 - \frac{\partial^2 E}{\partial v^2} = 0$$

Substituting equations (90) and (91) in equation (88) we have $K = \frac{1}{\rho_1\rho_2} = 0$, that is

(92) $$L = 0 \text{ or } N = 0,$$

and the surface (S) is therefore developable.

From equations (6), Table I, we have, on dividing by ρ_1^3 and ρ_2^3 respectively and substituting for $\frac{\sqrt{E}}{R_1}$ and $\frac{\sqrt{G}}{R_2}$ their values drawn from equations (87).

(93) $$\begin{cases} -\dfrac{1}{R_2}\cdot\dfrac{\partial R_2}{\partial v}\left\{\dfrac{1}{\rho_1\rho_2}-\dfrac{1}{\rho_1^2}\right\}=\dfrac{1}{\rho_1^2}\,\dfrac{\partial\rho_1}{\partial v}, \\ -\dfrac{1}{R_1}\cdot\dfrac{\partial R_1}{\partial u}\left\{\dfrac{1}{\rho_1\rho_2}-\dfrac{1}{\rho_2^2}\right\}=\dfrac{1}{\rho_2^2}\,\dfrac{\partial\rho_2}{\partial u}, \end{cases}$$

or since

$$\frac{1}{\rho_1\rho_2}=0$$

we have

(94) $$\begin{cases} \dfrac{1}{R_2}\,\dfrac{\partial R_2}{\partial v}=\dfrac{1}{\rho_1}\,\dfrac{\partial\rho_1}{\partial v} \\ \dfrac{1}{R_1}\,\dfrac{\partial R_1}{\partial u}=\dfrac{1}{\rho_2}\,\dfrac{\partial\rho_2}{\partial u}. \end{cases}$$

By aid of the relations in Table I equations (94) may be written

(95) $$\begin{cases} \dfrac{1}{R_2}\,\dfrac{\partial R_2}{\partial v}=\dfrac{1}{\rho_1}\,\dfrac{\partial\rho_1}{\partial v}=\dfrac{1}{h_1}\,\dfrac{\partial h_1}{\partial v}, \\ \dfrac{1}{R_1}\,\dfrac{\partial R_1}{\partial u}=\dfrac{1}{\rho_2}\,\dfrac{\partial\rho_2}{\partial u}=\dfrac{1}{h_2}\,\dfrac{\partial h_2}{\partial u}. \end{cases}$$

Eliminating $(\rho_2-\rho_1)$ from equations (6) Table I we have

(96) $$\frac{\sqrt{E}}{R_1}\cdot\frac{1}{\rho_1^2}\,\frac{\partial\rho_1}{\partial v}+\frac{\sqrt{G}}{R_2}\cdot\frac{1}{\rho_2^2}\cdot\frac{\partial\rho_2}{\partial u}=0.$$

Substituting from equations (87) and (95) in equation (96) we have

(97) $$\frac{1}{\rho_1}+\frac{1}{\rho_2}=0.$$

But when equations (95) are satisfied we have seen that $\frac{1}{\rho_1\rho_2}=0$, hence equa-

tion (97) indicates that

$$(98) \qquad \rho_1 = \infty \quad \rho_2 = \infty$$

which shows that the surface (S) is a plane. Neglecting this trivial solution of the problem we have the result that there is no surface for which the lines of curvature are reproduced as lines of curvature on the surfaces (S_1) and (S_2).

I shall now take the case where u and v are the parameters of the lines of curvature on the surface (S_1) but not on the surface (S_2). In this case we have

$$(99) \qquad \frac{\sqrt{E}}{R_1} = -\frac{1}{R_1} \cdot \frac{\partial R_1}{\partial u}.$$

If we divide equations (6), Table I, by ρ_2^3 and ρ_1^3 respectively and replace $\frac{1}{\rho_1\rho_2}$ by its value drawn from equation (89) with condition (98) included, we have, after some reductions,

$$(100) \qquad \frac{\sqrt{E}}{R_1}\left[\frac{1}{R_2\sqrt{G}}\left\{\frac{\sqrt{G}}{R_2} + \frac{1}{R_2}\frac{\partial R_2}{\partial v}\right\} + \frac{1}{\rho_2^2}\right] = -\frac{1}{\rho_1^2}$$ [illegible]

(01) $$\frac{\sqrt{G}}{R_2}\left[\frac{1}{R_1\sqrt{G}}\left\{\frac{\sqrt{G}}{R_2}+\frac{1}{R_2}\frac{\partial R_2}{\partial v}\right\}+\frac{1}{\rho_1^2}\right]=-\frac{1}{\rho_1^3}\frac{\partial \rho_1}{\partial v}.$$

From equation (01) on using equations (1) and (2), Table I, and the derivative of (2) with respect to v we derive the following expression

(02) $$\frac{\sqrt{G}}{R_2}=-\frac{1}{h_1}\frac{\partial h_1}{\partial v}=-\frac{1}{2E}\frac{\partial E}{\partial v}.$$

We also have from equations (3) and (4) Table I,

(03) $$\log h_1=\log \rho_1+\log\cos\varpi_1,$$

(04) $$\frac{1}{\rho_1}\frac{\partial \rho_1}{\partial v}=\frac{\sqrt{G}(\rho_2-\rho_1)}{\rho_1\rho_2}\tan\varpi_1,$$

(05) $$\tan\varpi_1=-\frac{\rho_1}{R_2},$$

(06) $$\frac{1}{h_1}\frac{\partial h_1}{\partial v}=-\frac{\sqrt{G}}{R_2\rho_2}(\rho_2-\rho_1)+\frac{\rho_1}{R_2}\frac{\partial \varpi_1}{\partial v}.$$

On substituting the value of $\frac{1}{h_1}\frac{\partial h_1}{\partial v}$, drawn from equation (06), in equation (02) we have

(07) $$\frac{1}{\rho_2}=-\frac{1}{\sqrt{G}}\frac{\partial \varpi_1}{\partial v}.$$

By means of equation (02) we may write equation (00) in the form

(08) $$-\frac{\sqrt{E}}{\sqrt{G}R_1R_2}\left\{-\frac{1}{h_1}\frac{\partial h_1}{\partial v}+\frac{1}{R_2}\frac{\partial R_2}{\partial v}\right\}=\frac{R_1}{\rho_2^3}\cdot\frac{R_1\frac{\partial \rho_2}{\partial u}-\rho_2\frac{\partial R_1}{\partial u}}{R_1^2}=\frac{1}{\rho_2^2}\cdot\frac{1}{\sin\varpi_2\cos\varpi_2}\cdot\frac{\partial \varpi_2}{\partial u}.$$

The left hand member of equation (108) is exactly

(109) $$\frac{\sqrt{E}}{\tau_1}\,\frac{1}{\rho_1\rho_2}$$

therefore we have

(110) $$\frac{1}{\rho_1\rho_2} = -\frac{1}{\rho_2}\cdot\frac{1}{\sin^2\varpi_2}\,\frac{1}{\sqrt{E}}\,\frac{\partial\varpi_2}{\partial u},$$

and finally

(111) $$\frac{1}{\rho_1} = -\frac{1}{\sin^2\varpi_2}\cdot\frac{1}{\sqrt{E}}\cdot\frac{\partial\varpi_2}{\partial u}.$$

It is easy to find an expression for $\frac{1}{\rho_1}$ in terms of ϖ_1, for we have from equation (5), Table I. and equation (17) on differentiating with regard to v and u respectively and equating the values of $\frac{\partial^2\varpi}{\partial u\,\partial v}$,

(112) $$\frac{\partial}{\partial v}\,\frac{\sqrt{E}}{\tau} + \frac{\partial}{\partial u}\,\frac{\sqrt{G}}{\rho_2} = 0.$$

On performing the indicated differentiation, using relations (1), Table I and substituting for $\frac{\partial\rho_2}{\partial u}$ its value drawn from equation (9), Table I we have after some reductions

(113) $$\frac{1}{\tau R_2} + \frac{1}{\sqrt{G}}\,\frac{1}{\tau_1^2}\cdot\frac{\partial\tau_1}{\partial v} + \frac{1}{\rho_1 R_1} = 0.$$

Now from equation (5), Table I, on differentiating it with respect to v, we find after some reductions

(114) $$\frac{1}{\sqrt{G}}\,\frac{1}{r_1^2}\cdot\frac{\partial r_1}{\partial v} = -\frac{1}{r_1 R_2} - \frac{1}{\sqrt{EG}}\cdot\frac{\partial^2\varpi_1}{\partial u\,\partial v}.$$

therefore we have finally

(115) $$\frac{1}{\rho_1} = \frac{R_1}{\sqrt{EG}}\cdot\frac{\partial^2\varpi_1}{\partial u\,\partial v}.$$

On substituting in the first of equations (6), Table I, the value of $\frac{1}{\sqrt{E}}$ drawn from the condition (9) we find the following relation

(116) $$\frac{1}{R_1}\,\frac{\partial R_1}{\partial u} = \frac{1}{2G}\,\frac{\partial G}{\partial u};$$

which on substituting in the second of equations (6), Table I gives the following,

(117) $$\frac{1}{\rho_2}\,\frac{\partial \rho_2}{\partial u} = \frac{1}{R_1}\,\frac{\partial R_1}{\partial u}\left(1-\frac{\rho_2}{\rho_1}\right).$$

If now the supposition

(118) $$\rho_2 = \pm R_1\sqrt{-1}$$

be made, equation (117) becomes

(119) $$\frac{\rho_2}{\rho_1} = 0, \quad \rho_1 = \infty,$$

which indicates a developable surface. In this last case I have not as yet been

able to interpret the results so as to get the general solution of the problem, but hope to be able to do so at some future time.

II

I shall now discuss the special example spoken of in the Introduction; that is the surface (S_1): the one nappe of the focal surface of the congruence formed by the tangents to the u lines of the ellipsoid (S), which itself forms the other nappe of the focal surface.

The coordinates of a point $P(x, y, z)$ on the ellipsoid (S) are given in terms of the parameters u and v as follows.

$$(1)\qquad \begin{cases} x = \sqrt{-\frac{a^2\lambda}{\delta}\cdot \overline{a^2+u}\cdot\overline{a^2+v}}\,, \\ y = \sqrt{-\frac{b^2\mu}{\delta}\cdot \overline{b^2+u}\cdot\overline{b^2+v}}\,, \\ z = \sqrt{-\frac{c^2\nu}{\delta}\cdot \overline{c^2+u}\cdot\overline{c^2+v}}\,; \end{cases}$$

where

$$(2)\qquad \lambda = b^2-c^2,\quad \mu = c^2-a^2,\quad \nu = a^2-b^2,\quad \delta = \lambda\mu\nu$$

The fundamental coefficients E, F, G, L, M, N; the direction cosines of the normal to the surface (S) and the tangents to the u and v lines respectively; the radii of curvature of the principal sections; the radii of geodesic curvature; the torsion and the radii of curvature of the u and v lines respectively; the sine, cosine and tangent of the angles between the normal to the surface (S) and the principal normal to the u and v lines respectively, are, for convenience in future reference, given in the following table which I shall call Table II.

(3) $$E = \frac{u(u-v)}{4.a^2+u.b^2+u.c^2+u} = \frac{u(u-v)}{4\,U_1};$$

(4) $$G = \frac{v(v-u)}{4.a^2+v.b^2+v.c^2+v} = \frac{v(v-u)}{4\,V_1};$$

(5) $$F = M = 0;$$

(6) $$L = \frac{abc\,(v-u)}{4U_1\sqrt{uv}};$$

(7) $$N = \frac{abc\,(u-v)}{4V_1\sqrt{uv}}.$$

$$\tag{8}\left\{\begin{aligned}\alpha &= \sqrt{-\frac{b^2c^2\cdot a^2+u\cdot a^2+v}{\mu\nu\; uv}},\\ \beta &= \sqrt{-\frac{c^2a^2\cdot b^2+u\cdot b^2+v}{\nu\lambda\; uv}},\\ \gamma &= \sqrt{-\frac{a^2b^2\cdot c^2+u\cdot c^2+v}{\lambda\mu\cdot uv}};\end{aligned}\right.$$

$$\tag{9}\left\{\begin{aligned}\alpha' &= \sqrt{-\frac{a^2\lambda\, U_1\cdot a^2+v}{\delta u(u-v)\, a^2+u}},\\ \beta' &= \sqrt{-\frac{b^2\mu\, U_1\cdot b^2+v}{\delta u(u-v)\cdot b^2+u}},\\ \gamma' &= \sqrt{-\frac{c^2\nu\, U_1\cdot c^2+v}{\delta u(u-v)\; c^2+u}};\end{aligned}\right.$$

$$\tag{10}\left\{\begin{aligned}\alpha'' &= \sqrt{-\frac{a^2\lambda\, V_1\cdot a^2+u}{\delta v(v-u)\, a^2+v}},\\ \beta'' &= \sqrt{-\frac{b^2\mu\, V_1\cdot b^2+u}{\delta v(v-u)\cdot b^2+v}},\\ \gamma'' &= \sqrt{-\frac{c^2\nu\, V_1\cdot c^2+u}{\delta v(v-u)\cdot c^2+v}};\end{aligned}\right.$$

$$\tag{11}\rho_1 = -\frac{u\sqrt{uv}}{abc};$$

$$\tag{12}\rho_2 = -\frac{v\sqrt{uv}}{abc}.$$

$$\tag{13}R_1 = -\sqrt{\frac{u(u-v)^3}{U_1}};$$

$$\tag{14}R_2 = -\sqrt{\frac{v(v-u)^3}{V_1}};$$

$$\tag{15}h_1 = \sqrt{\frac{u^3v\,(v-u)^3}{u^3V_1+a^2b^2c^2(v-u)^3}};$$

$$\tag{16}h_2 = \sqrt{\frac{v^3u\,(u-v)^3}{v^3U_1+a^2b^2c^2(u-v)^3}};$$

$$\tag{17}\left\{\begin{aligned}\sin\varpi_1 &= \sqrt{\frac{u^3V_1}{u^3V_1+a^2b^2c^2(v-u)^3}},\\ \cos\varpi_1 &= -\sqrt{\frac{a^2b^2c^2(v-u)^3}{u^3V_1+a^2b^2c^2(v-u)^3}},\\ \tan\varpi_1 &= -\sqrt{\frac{u^3V_1}{a^2b^2c^2(v-u)^3}}.\end{aligned}\right.$$

$$(18)\qquad \begin{cases} \sin \varpi_2 = \sqrt{\dfrac{v^3U_1}{v^3U_1 + a^2b^2c^2(u-v)^3}}, \\ \cos \varpi_2 = \sqrt{\dfrac{a^2b^2c^2(u-v)^3}{v^3U_1 + a^2b^2c^2(u-v)^3}}, \\ \tan \varpi_2 = -\sqrt{\dfrac{v^3U_1}{a^2b^2c^2(u-v)^3}}; \end{cases}$$

$$(19)\qquad \frac{1}{\tau_1} = -\frac{3abc\,v\sqrt{-U_1V_1}}{u^3V_1 + a^2b^2c^2(v-u)^3};$$

$$(20)\qquad \frac{1}{\tau_2} = -\frac{3abc\,v\sqrt{-U_1V_1}}{v^3U_1 + a^2b^2c^2(u-v)^3};$$

$$(21)\qquad EG = -\frac{uv(u-v)^2}{16U_1V_1};$$

$$(22)\qquad LN = -\frac{a^2b^2c^2(u-v)^2}{16U_1V_1uv}.$$

The coordinates of a point $P(X_1, Y_1, Z_1)$ on the surface (S_1) are given by the equations

$$(23)\qquad \begin{cases} X_1 = \sqrt{-\dfrac{a^2\lambda(a^2+v)^3}{\delta\ a^2+u}}, \\ Y_1 = \sqrt{-\dfrac{b^2\mu(b^2+v)^3}{\delta\ b^2+u}} \\ Z_1 = \sqrt{-\dfrac{c^2\nu(c^2+v)^3}{\delta\ c^2+u}}. \end{cases}$$

The equation of the surface (S_1) in Cartesian coordinates is derived by eliminating u and v between the equations (23) it becomes after certain reductions

$$(24)\qquad \left[\Sigma a^2\lambda^2Y_1^2Z_1^2 - \Sigma b^2c^2\lambda^2X_1^2\right]^3 + 27\lambda^2\mu^2\nu^2X_1^2Y_1^2Z_1^2\left[\Sigma b^2c^2X_1^2 - a^2b^2c^2\right]^2 = 0.$$

It is therefore a surface of the twelfth order.

I shall now proceed to find the fundamental coefficients and the direction cosines of the normal to the surface (S_1). Using the formulae derived in the general case and substituting in them the special values drawn from Table II, we have, after some reductions,

(25) $$E_1 = \frac{E}{h_1^2}\left\{h_1^2 U^2 + R_1^2\right\} = \frac{1}{E}\left\{\frac{\partial(u-v)E}{\partial u}\right\}^2 + 4\frac{(u-v)^2E^2}{h_1^2};$$

(26) $$F_1 = \sqrt{E}\,U\cdot\frac{\partial R_1}{\partial v} = -3\,\frac{\partial(u-v)E}{\partial u};$$

(27) $$G_1 = \left(\frac{\partial R_1}{\partial v}\right)^2 = 9E;$$

(28) $$H_1^2 = \frac{R_1^2E}{h_1^2}\cdot\left(\frac{\partial R_1}{\partial v}\right)^2 = 36\,\frac{(u-v)^2E^3}{h_1^2};$$

(29) $$\left\{\begin{aligned} \alpha_1 &= h_1\left(\frac{\alpha'}{\rho_1} - \frac{\alpha}{R_2}\right) = \frac{h_1\alpha}{R_2}\left\{1 + \frac{a^2(u-v)}{u(a^2+v)}\right\}, \\ \beta_1 &= h_1\left(\frac{\beta''}{\rho_1} - \frac{\beta}{R_2}\right) = \frac{h_1\beta}{R_2}\left\{1 + \frac{b^2(u-v)}{u(b^2+v)}\right\}, \\ \gamma_1 &= h_1\left(\frac{\gamma'}{\rho_1} - \frac{\gamma}{R_2}\right) = \frac{h_1\gamma}{R_2}\left\{1 + \frac{c^2(u-v)}{u(c^2+v)}\right\}. \end{aligned}\right.$$

(30) $$L_1 = -\frac{E R_1}{h_1\tau_1} = -\frac{3}{2}\cdot\frac{h_1 v E}{\rho_1\sqrt{G}\,u(u-v)};$$

(31) $$M_1 = 0;$$

(32) $$N_1 = -\frac{\sqrt{G}\,h_1}{R_1\rho_1}\cdot\frac{\partial R_1}{\partial v} = \frac{3}{2}\,\frac{\sqrt{G}\,h_1}{\rho_1(u-v)};$$

(33) $$L_1N_1 = \frac{E\sqrt{G}}{\rho_1\tau_1}\cdot\frac{\partial R_1}{\partial v} = -\frac{9}{4}\,\frac{h_1^2 v E}{\rho_1^2 u(u-v)^2}.$$

We have now for the measure of curva-

ture at a point $P(X_1, Y_1, Z_1)$ on the surface (S_1) the following expression

(34) $$K_1 = \frac{1}{\rho_{11}\rho_{12}} = \frac{L_1 N_1}{E_1 G_1 - F_1^2} = -\frac{1}{9}\,\frac{h^4 v}{E^2 \rho_1^2\, u(u-v)^4},$$

which we see on inspection is negative and therefore the surface (S_1) has real asymptotic lines.

To determine the radii of curvature of the principal sections we have the following equation,

(35) $$L_1 N_1 \rho'^2 - (G_1 L_1 + E_1 N_1)\rho' + E_1 G_1 - F_1^2 = 0,$$

which gives on solving

(36) $$\rho_{11}, \rho_{12} = \frac{G_1 L_1 + E_1 N_1 \pm \sqrt{(G_1 L_1 - E_1 N_1)^2 + 4F_1^2 L_1 N_1}}{2 L_1 N_1}.$$

This last equation becomes, on substituting the values of the different quantities drawn from equations (25) to (33) inclusive,

(37) $$\rho_{11}, \rho_{12} = -\frac{\rho_1 \sqrt{G(u-v)u}}{3h_1 v E^2}\Bigg[\left\{\frac{\partial (u-v)E}{\partial u}\right\}^2 + E^2\left\{4E\frac{(u-v)^2}{h_1^2} + 9\frac{V_1}{U_1}\right\} \pm \sqrt{\left[\left\{\frac{\partial (u-v)E}{\partial u}\right\}^2 + E^2\left\{4E\frac{(u-v)^2}{h_1^2} + 9\frac{V_1}{U_1}\right\}\right]^2 + 144\frac{E^4 v(u-v)}{h_1^4 G u}}\Bigg],$$

which also shows that ρ_{11}, ρ_{12} have op-

posite signs.

The asymptotic lines on the surface (S_1) are given by the following equation,

$$L_1\,du^2 + N_1\,dv^2 = 0, \tag{38}$$

which, on substituting for L_1 and N_1 their values drawn from equations (30) and (32), becomes

$$\frac{du^2}{U_1} + \frac{dv^2}{V_1} = 0. \tag{39}$$

This gives for the determination of the asymptotic lines the two differential equations of the first order and degree

$$\left\{\begin{aligned} \frac{du}{\sqrt{U_1}} + i\frac{dv}{\sqrt{V_1}} = 0,\\ \frac{du}{\sqrt{U_1}} - i\frac{dv}{\sqrt{V_1}} = 0, \end{aligned}\right. \tag{40}$$

which we immediately recognize as Euler's equation.

The differential equation of the lines of curvature becomes, in this case where u and v are the parameters of conjugate lines,

(41) $$F_1 L_1\, du^2 + (G_1 L_1 - E_1 N_1)\, du\, dv - F_1 N_1\, dv^2 = 0,$$

which becomes on making the proper substitutions and reducing,

(42) $$\frac{du^2}{U_1} + \frac{1}{12V_1}\left[\frac{1}{E}\left\{\frac{\partial(u-v)E}{\partial u}\right\} + \frac{64E^2}{u^3 v(u-v)}\left\{\frac{\partial(u-v)E}{\partial u}\right\}^{-1}\left\{(u+9v)u^2V_1 + u^2v^2(v-u)^3\right\}\right]du\,dv - \frac{dv^2}{V_1} = 0.$$

The differential equations of the surfaces (S) and (S,) reduce to the following forms.

In the case of the Ellipsoid (S) we have the well known equation

(43) $$\frac{\partial^2\theta}{\partial u\,\partial v} + \frac{1}{2(u-v)}\frac{\partial\theta}{\partial u} - \frac{1}{2(u-v)}\frac{\partial\theta}{\partial v} = 0,$$

which has for particular solutions

(44) $$\theta = x, y, z;$$

and for invariants

(45) $$h = k = -\frac{3}{4(u-v)^2}.$$

In the case of the surface (S,) we have the equation

$$\frac{\partial^2\sigma}{\partial u\,\partial v} + \frac{3}{2(u-v)}\frac{\partial\sigma}{\partial u} + \frac{1}{2(u-v)}\frac{\partial\sigma}{\partial v} = 0,$$

which has for particular solutions

$$\sigma = X_1, Y_1, Z_1;$$

and for invariants

$$(48)\qquad \begin{cases} h_1 = -\dfrac{3}{4(u-v)^2}, \\ k_1 = \dfrac{5}{4(u-v)^2}. \end{cases}$$

The invariants h_1 and k_1 are not equal. The u and v lines do not form an isothermal system on the surface (S_1), but they are conjugate as equation (46) shows.

I shall now consider some of the properties of the surface (S_1). First it must be noticed that, while u and v when taken separately may each vary from $-a^2$ to $-c^2$, when taken together they may only vary between the limits $-a^2, -b^2$; $-b^2, -c^2$; respectively.

We have as has already been shown, the equation of the surface (S_1) in terms of the parameters u and v; if now we place $v = -b^2$ we get the section of the surface (S_1) by the plane $y = 0$, which is

$$(49)\qquad \begin{cases} X_1 = \sqrt{-\frac{a^2\nu^2}{\mu \cdot a^2+u}}\,, \\ Y_1 = 0, \\ Z_1 = \sqrt{\frac{c^2\lambda^2}{\mu \cdot c^2+u}}\,, \end{cases}$$

or on eliminating u between equations (49)

$$(50)\qquad \mu^2 X_1^2 Z_1^2 - c^2\lambda^2 X_1^2 - a^2\nu^2 Z_1^2 = 0.$$

This curve denoted in equation (49) or (50) is a cuspidal line of the surface (S_1) as I shall proceed to show. The equation of any tangent line to the surface (S) at the point $P(x,y,z)$ is

$$(51)\qquad \frac{\xi - x}{\frac{dx}{ds}} = \frac{\eta - y}{\frac{dy}{ds}} = \frac{\zeta - z}{\frac{dz}{ds}}.$$

In the plane $y=0$, that is the point $P(x,0,z)$ for which we have $v=-b^2$, it becomes

$$(52)\qquad \frac{\xi - x}{\frac{\partial x}{\partial u}} = \frac{\zeta - z}{\frac{\partial z}{\partial u}}\,,\quad y=0.$$

The consecutive tangents above and near the point $P(x,0,z)$ that is at the points $P(x, \pm\Delta y = y', z)$ for which we have $v=-b^2 \pm \kappa^2$, have for their equations

$$(53)\qquad \frac{\xi - x}{\frac{\partial x}{\partial u}} = \frac{\eta - y'}{\left(\frac{\partial y}{\partial u}\right)_{y=y'}} = \frac{\zeta - z}{\frac{\partial z}{\partial u}}.$$

The intersection of the lines represented by equations (5·2) and (5·3) gives for the coordinates of the point of intersection $P(\xi,\eta,\zeta)$

$$(5\cdot4)\qquad \begin{cases} \xi' = x - y\,\dfrac{\frac{\partial x}{\partial u}}{\left(\frac{\partial y}{\partial u}\right)_{y=y}} \\ \eta' = 0, \\ \zeta' = z - y'\,\dfrac{\frac{\partial z}{\partial u}}{\left(\frac{\partial y}{\partial u}\right)_{y=y}}\,. \end{cases}$$

But we have

$$(5\cdot5)\qquad \begin{cases} \pm dy = y' = \sqrt{-\frac{b^2\mu}{\delta}\,(b^2+u)\cdot(\pm k^2)}\,, \\ \frac{\partial x}{\partial u} = \frac{1}{2}\sqrt{-\frac{a^2\lambda\,(\nu\pm k^2)}{\delta\cdot a^2+u}}\,, \\ \left(\frac{\partial y}{\partial u}\right)_{y=y'} = \frac{1}{2}\sqrt{-\frac{b^2\mu\,(\pm k^2)}{\delta\cdot b^2+u}}\,, \\ \frac{\partial z}{\partial u} = \frac{1}{2}\sqrt{-\frac{c^2\nu\,(-\lambda\pm k^2)}{\delta\cdot c^2+u}}\,. \end{cases}$$

Therefore we have on substituting & reducing.

$$(5\cdot6)\qquad \begin{cases} \xi' = \sqrt{-\frac{a^2\nu^2}{\mu\cdot a^2+u}} \mp \frac{1}{2}\,k^2\cdot b^2+u\cdot\sqrt{-\frac{a^2\lambda}{\nu\cdot a^2+u}}\,, \\ \eta' = 0, \\ \zeta' = \sqrt{\frac{c^2\lambda^2}{\mu\cdot c^2+u}} \mp \frac{1}{2}\,k^2\cdot b^2+u\cdot\sqrt{\frac{c^2\mu}{\lambda\cdot c^2+u}}\,. \end{cases}$$

Since k^2 is an infinitesimal of the second order it may be dropped and we have finally

$$(5\cdot7)\qquad \begin{cases} \xi' = \sqrt{-\frac{a^2\nu^2}{\mu\cdot a^2+u}}\,, \\ \eta' = 0, \\ \zeta' = \sqrt{\frac{c^2\lambda^2}{\mu\cdot c^2+u}}\,; \end{cases}$$

which upon inspection of equations (49) we see are identically the coordinates of the section of the surface (S_1) by the plane $y=0$. Therefore that section is a cuspidal line of the surface (S_1), for it is the locus of the intersection of three consecutive tangents on the surface (S).

If the equation of the surface (S_1) given in point coordinates had been considered, we should have seen immediately that the section by the plane $y=0$ was a cuspidal line, for we should have had for its equation

$$(58) \qquad \left[\mu^2 X_1^2 Z_1^2 - c^2\lambda^2 X_1^2 - a^2\nu^2 Z_1^2\right]^3 = 0 .$$

I shall now consider the section by the plane $y=0$ its equation is, as we have seen,

$$(59) \qquad \mu^2 X_1^2 Z_1^2 - c^2\lambda^2 X_1^2 - a^2\nu^2 Z_1^2 = 0 .$$

It may be written in the form

(60) $$\mu^2 X_1^2 Z_1^2 - t^2 - \{c^2\lambda^2 X_1^2 + a^2\nu^2 Z_1^2 - t^2\} = 0,$$

where t is a variable parameter. We now see that the section is the locus of the intersection of the conjugate equilateral hyperbolas,

(61) $$\begin{cases} \mu X_1 Z_1 - t = 0, \\ \mu X_1 Z_1 + t = 0; \end{cases}$$

with the ellipse

(62) $$c^2\lambda^2 X_1^2 + a^2\nu^2 Z_1^2 - t^2 = 0.$$

Since the curve is symmetrical with respect to the axes of X and Z we need only consider the intersection of one of the hyperbolas (61) with the ellipse (62). We find on substituting in the first of the equations (61) the value of X_1 drawn from equation (62) and reducing, that

(63) $$\begin{cases} Z_1^2 = \dfrac{t^2}{2a^2\nu^2}\left\{1 \pm \sqrt{1 - \dfrac{4a^2c^2\lambda^2\nu^2}{t^2\mu^2}}\right\}, \\ Z_1 = \pm \dfrac{t}{\sqrt{2}\,a\nu}\sqrt{1 \pm \sqrt{1 - 4\dfrac{a^2c^2\lambda^2\nu^2}{t^2\mu^2}}}, \end{cases}$$

and therefore

(64) $$X_1 = \pm \frac{t}{\sqrt{2}\,c\lambda}\sqrt{1 \mp \sqrt{1-4\frac{a^2c^2\lambda^2\nu^2}{t^2\mu^2}}}\,.$$

The real and imaginary values of X_1, Z_1 come from the consideration of the radical which occurs in the value of Z_1^2.

If we have

(65) $$\begin{cases} (a)\quad t^2 < 4\frac{a^2c^2\lambda^2\nu^2}{\mu^2}, & Z_1, X_1, \text{ both imaginary;} \\ (b)\quad t^2 = 4\frac{a^2c^2\lambda^2\nu^2}{\mu^2}, & Z_1, X_1, \text{ both real; values equal, signs opposite,} \\ (c)\quad t^2 > 4\frac{a^2c^2\lambda^2\nu^2}{\mu^2}, & Z_1, X_1, \text{ both real.} \end{cases}$$

Condition (b) gives the values

(66) $$\begin{cases} X_1 = \pm\sqrt{2}\,\frac{a\nu}{\mu}, \\ Z_1 = \pm\sqrt{2}\,\frac{c\lambda}{\mu}; \end{cases}$$

the coordinates of the points at which the hyperbolae are tangent to the ellipse and the first real points on the curve. On consideration of the equation (59) or (60) we see that the origin is a conjugate point on the curve. We have for the equation of the imaginary tangents at the origin

(67) $$c^2\lambda^2X_1^2 + a^2\nu^2Z_1^2 = 0.$$

Any line through the origin cuts the curve in two real points.

Making equation (59) homogeneous we find for the equation of the asymptotes

(68) $$X_1^2 Z_1^2 - \left\{\frac{c^2\lambda^2 X_1^2 + a^2\nu^2 Z_1^2}{\mu^2}\right\} W_1^2 = 0;$$

therefore X_1, Z_1 are the asymptotes to the curve. There are two double points at infinity and W_1 is the tangent to them.

The coordinates of the real umbilics in the plane $y=0$ are derived for the ellipsoid by making $u=v=-b^2$, and are

(69) $$\begin{cases} X = \pm\sqrt{-\frac{a^2\nu}{\mu}}, \\ Y = 0, \\ Z = \pm\sqrt{-\frac{c^2\lambda}{\mu}}. \end{cases}$$

These values will satisfy equation (59) therefore the surface (S_1) passes through the umbilics of the Ellipsoid (S).

The sections of the surface (S_1) in the planes $x=0$, $z=0$, and the plane at infinity

are derived in precisely the same manner as for the plane $u=0$ in placing

$$v = -a^2, -c^2, \infty^2;$$

respectively, which give for the different sections the equations,

$$(70)\quad \begin{cases} X_1 = 0, \\ Y_1 = \sqrt{\frac{b^2\nu^2}{\lambda . b^2+u}}, \\ Z_1 = \sqrt{-\frac{c^2\mu^2}{\lambda . c^2+u}}; \\ \lambda^2 Y_1^2 Z_1^2 - c^2\mu^2 Y_1^2 - b^2\nu^2 Z_1^2 = 0; \end{cases}$$

$$(71)\quad \begin{cases} X_1 = \sqrt{\frac{a^2\mu^2}{\nu . a^2+u}}, \\ Y_1 = \sqrt{-\frac{b^2\lambda^2}{\nu\, b^2+u}}, \\ Z_1 = 0; \\ \nu^2 X_1^2 Y_1^2 - b^2\lambda^2 X_1^2 - a^2\mu^2 Y_1^2 = 0 \end{cases}$$

$$(72)\quad \begin{cases} X_1 : Y_1 : Z_1 = \sqrt{-\frac{a^2\lambda}{\delta . a^2+u}} : \sqrt{-\frac{b^2\mu}{\delta . b^2+u}} : \sqrt{-\frac{c^2\nu}{\delta . c^2+u}}, \\ a^2\lambda^2 Y_1^2 Z_1^2 + b^2\mu^2 Z_1^2 X_1^2 + c^2\nu^2 X_1^2 Y_1^2 = 0. \end{cases}$$

They are each a cuspidal line on the surface (S_1) and pass through the imaginary umbilics of the Ellipsoid (S) in their respective planes. The coor-

dinates of these umbilics are derived by making

$$u = v = -a^2,\ -c^2,\ \infty^2$$

respectively, and are given by the expressions

(73)
$$\begin{cases} X = 0, \\ Y = \pm\sqrt{-\frac{b^2\nu}{\lambda}}, \\ Z = \pm\sqrt{-\frac{c^2\mu}{\lambda}}; \end{cases}$$

(74)
$$\begin{cases} X = \pm\sqrt{-\frac{a^2\mu}{\nu}}, \\ Y = \pm\sqrt{-\frac{b^2\lambda}{\nu}}, \\ Z = 0; \end{cases}$$

(75)
$$X : Y : Z = \sqrt{-\frac{a^2\lambda}{\delta}} : \sqrt{-\frac{b^2\mu}{\delta}} : \sqrt{-\frac{c^2\nu}{\delta}}.$$

The point $(0, 0, 0)$ on the surface (S_1) corresponds to the points at infinity on the ellipsoid (S), for we have, for $u = \infty^2$,

(76)
$$R_1 = -\infty;\ X_1 = 0,\ Y_1 = 0,\ Z_1 = 0.$$

The surfaces (S) and (S_1) do not intersect at all but are tangent to each other along the curve for which $u = v$, and

Therefore

$$X_1 = x,\ Y_1 = y,\ Z_1 = z;$$

which will now be demonstrated. The tangent plane at any point of the Ellipsoid (S) is given by the equation

$$(77)\qquad (\xi - x)\alpha + (\eta - y)\beta + (\zeta - z)\gamma = 0.$$

The tangent plane to the surface (S_1) at any point is given by the equation

$$(78)\qquad (\xi - X_1)\alpha_1 + (\eta - Y_1)\beta_1 + (\zeta - Z_1)\gamma_1 = 0,$$

or, after substituting the values of $\alpha_1, \beta_1, \gamma_1$ drawn from expressions (29) and making certain reductions,

$$(79)\qquad (\xi - X_1)\alpha\left\{1 + \frac{a^2(u-v)}{u(a^2+v)}\right\} + (\eta - Y_1)\beta\left\{1 + \frac{b^2(u-v)}{u(b^2+v)}\right\} + (\zeta - Z_1)\gamma\left\{1 + \frac{c^2(u-v)}{u(c^2+v)}\right\} = 0.$$

In consideration of the points where $u = v$ and $X_1 = x$, $Y_1 = y$, $Z_1 = z$, we see that the equations of the tangent planes to the two surfaces (S) and (S_1) are identical at those points and are represented by the following equation

(80) $$(\xi - X)\alpha + (\eta - Y)\beta + (\zeta - Z)\gamma = 0.$$

On examining the expressions for the direction cosines of the normals to the surfaces (S) and (S₁) we see that, when $u = v$, we have

(81) $$\alpha_1 = -\alpha,\ \beta_1 = -\beta,\ \gamma_1 = -\gamma,$$

which indicates that the curvature of the surfaces (S) and (S₁) is in opposite directions at these points. The only real points on the curve where $u = v$ are the real umbilics in the plane $y = 0$, therefore the surfaces (S) and (S₁) are in reality tangent only at that point.

The above may be derived in an analogous manner for the surface (S_2); or by simply interchanging $\underline{u}$ and $\underline{v}$ and observing the changes arising out of that interchange.

There are I suppose further properties

belonging to the surfaces (S), (S_1), and (S_2), and relations connecting them but as yet I have not been able to derive them.

Biographical Sketch.

The author, Thomas Hardy Taliaferro, was born in Jacksonville, Florida, March 22nd, 1871. His elementary education was, for the most part, received in Mr. Gatewood's School in Norfolk, Virginia. He entered the Virginia Military Institute, Lexington Virginia, as a cadet in September 1886 and upon graduation in June 1890 was appointed Assistant Professor of Mathematics, which position he held until June 1891 when he accepted the position of Professor of Mathematics in the Missouri Military Academy, Mexico, Missouri. He entered the University in the fall of 1892, as a candidate for the degree of Doctor of Philosophy, selecting Mathematics as his principal, and Physics and

Astronomy as his first and second subordinate subjects examination. He held a Virginia scholarship for three years. During the past year he has held the Fellowship in Mathematics